# ON THE SIZE OF K-FOLD SUM AND PRODUCT SETS OF INTEGERS

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## Section 1. Statement of the result and outline of argument

We prove the following

**Theorem.** *For all $b \in \mathbb{Z}_+$, there is $k = k(b) \in \mathbb{Z}_+$ such that if $A \subset \mathbb{Z}$ is any finite set, with $|A| = N \geq 2$, then either*

$$|kA| = |\underbrace{A + \cdots + A}_{k}| > N^b, \tag{1.1}$$

*or*

$$|A^{(k)}| = |\underbrace{A \times \cdots \times A}_{k}| > N^b. \tag{1.2}$$

This result is one more contribution to a line of research initiated in the seminal paper [E-S] of Erdös and Szemerédi on sum and product sets. They conjectured that if $A \subset \mathbb{Z}$, with $|A| = N$, then

$$|2A| + |A^{(2)}| > c_\varepsilon N^{2-\varepsilon}, \quad \text{for all } \varepsilon > 0, \tag{i}$$

and more generally, for $k \geq 2$ an integer,

$$|kA| + |A^{(k)}| > c_\varepsilon N^{k-\varepsilon}, \quad \text{for all } \varepsilon > 0, \tag{ii}$$

Already (i) is open. Recent advances were achieved by G. Elekes [E] and J. Solymosi [So] using the Szemerédi-Trotter theorem in incidence geometry. It was shown in [E] that

$$|2A| + |A^{(2)}| > cN^{\frac{5}{4}},$$

and in [So] a small further improvement

$$|2A| + |A^{(2)}| > cN^{\frac{14}{11}-\varepsilon}.$$

Again, using the Szemerédi-Trotter theorem, Elekes-Nathanson-Ruzsa showed that

$$|kA| \cdot |A^{(k)}| > c|A|^{3-2^{1-k}}.$$

(See [E-N-R].) The theorem proved in this paper answers affirmatively a problem posed in their paper and provides further progress towards (ii). This same issue was also brought up independently by S. Konjagin (private communication) and was motivated by questions related to Gauss-sums (see [B-K]). Some further refinements of the statement of the theorem appear in Section 6 (Remarks) at the end of the paper.

The general strategy of our proof bares resemblance with [Ch] in several ways. Thus we assume $|A^{(k)}|$ 'small' and prove that then $|kA|$ has to be large. However, 'smallness' of $|A \cdot A|$ in [Ch] is the assumption

$$|A \cdot A| < K|A| \tag{1.4}$$

with $K$ a constant (a condition much too restrictive for our purpose).

If (1.4) holds, it is shown in [Ch] that

$$|A + A| > c(K)|A|^2 \tag{1.5}$$

and more generally

$$|hA| > c(K, h)|A|^h. \tag{1.6}$$

Let us briefly recall the approach.

Consider the map given by prime factorization

$$\mathbb{Z}_+ \longrightarrow \mathcal{R} = \prod_p \mathbb{Z}_{\geq 0}$$

$$n = \prod_p p^{\alpha_p} \longrightarrow \alpha = (\alpha_p)_p$$

where $p$ runs in the set $\mathcal{P}$ of primes.

The set $A$ is mapped to $\mathcal{A} \subset \mathcal{R}$ satisfying by (1.4)

$$|2\mathcal{A}| < K|\mathcal{A}|. \tag{1.4'}$$

Freiman's lemma implies then that $\dim \mathcal{A} < K$ (where 'dim' refers to the dimension of the smallest vector space containing $\mathcal{A}$). Hence there is a subset $I \subset \mathcal{P}, |I| < K$ such that the restriction $\pi_I$ is one-to-one restricted to $\mathcal{A}$. Harmonic analysis implies then that

$$\lambda_q(A) < (Cq)^K \tag{1.7}$$

for an absolute constant $C$, and for all $q > 2$. By $\lambda_q(A)$, we mean the $\Lambda_q$-constant of the finite set $A \subset \mathbb{Z}$, defined by

$$\lambda_q(A) = \max \left\| \sum_{n \in A} c_n \; e^{2\pi i n \theta} \right\|_{L^q(\mathbb{T})} \tag{1.8}$$

where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and the max is taken over all sequences $(c_n)_{n \in A}$ with $\left(\sum c_n^2\right)^{1/2} \leq 1$. See [Ru] and [Ki], for more details. As is [Ch], we will also make here crucial use of certain moment inequalities involving $\lambda_q$-constant of certain specific sets of integers. More specifically, we will use the following general inequality, from which (1.7) is derived.

**Proposition 1.** *(see [Ch]).*

*Let $p_1, \dots, p_k$ be distinct primes and associate to each $\alpha = (\alpha_1, \dots, \alpha_k) \in (\mathbb{Z}_{\geq 0})^k$ a trigonometric polynomial $F_\alpha$ on $\mathbb{T}$ such that*

$$(n, p) = 1, \text{ for all } n \in \operatorname{supp} \widehat{F}_\alpha, \text{ and for all } p \in \mathcal{P}_0.$$

*Then, for any moment $q \geq 2$*

$$\left\| \sum_\alpha F_\alpha(p_1^{\alpha_1} \cdots p_k^{\alpha_k} \theta) \right\|_q < (Cq)^k \left( \sum \|F_\alpha\|_q^2 \right)^{1/2}. \tag{1.9}$$

Thus (1.7) follows from (1.9) taking $F_\alpha(\theta) = e^{2\pi i \theta}$ and $\{p_1, \cdots, p_k\} = I \subset \mathcal{P}$.

Denoting for $h \geq 2$

$$r_h(n; A) = |\{(x_1, \dots, x_h) \in A^h | n = x_1 + \cdots + x_h\}|$$

A simple application of Parseval's identity gives

$$\sum_{n \in hA} r_h(n; A)^2 \leq \lambda_{2h}(A)^{2h} \cdot |A|^h$$

and using Cauchy-Schwartz inequality on $\sum_{n \in hA} r_h(n; A)$, it follows that

$$|hA| \geq \frac{|A|^h}{\lambda_{2h}(A)^{2h}}. \tag{1.10}$$

Thus we obtain (1.6) with

$$c(K, h) > (Ch)^{-2hK}.$$

Obviously, this statement has no interest unless $K \ll \log |A|$.

The main point in what follows is to be able to carry some of the preceding analysis under a much weaker assumption $K < |A|^\varepsilon$, $\varepsilon$ small. We will prove the following statement

**Proposition 2.** *Given $\gamma > 0$ and $q > 2$, there is a constant $\Lambda = \Lambda(\gamma, q)$ such that if $A \subset \mathbb{Z}$ is a finite set, $|A| = N, |A \cdot A| < KN$, then*

$$\lambda_q(A) < K^\Lambda N^\gamma \tag{1.11}$$

Thus fixing $q$, Proposition 2 provides already nontrivial information assuming $K < N^\delta$, with $\delta > 0$ sufficiently small.

Assuming Proposition 2, let us derive the Theorem. We may assume that $A \subset \mathbb{Z}_+$ to simplify the situation.

Fix $b$ and assume (1.2) fails for some large $k = 2^\ell$ (to be specified). Hence, passing to $\mathcal{A}$

$$|k\mathcal{A}| < N^b$$

$$\frac{|2^\ell \mathcal{A}|}{|2^{\ell-1}\mathcal{A}|} \frac{|2^{\ell-1}\mathcal{A}|}{|2^{\ell-2}\mathcal{A}|} \cdots \frac{|2\mathcal{A}|}{|\mathcal{A}|} < N^{b-1}$$

and we may find $k_0 = 2^{\ell_0}$ such that

$$\frac{|2k_0\mathcal{A}|}{|k_0\mathcal{A}|} < N^{\frac{b-1}{\ell}}. \tag{1.12}$$

Denote $\mathcal{B} = k_0\mathcal{A} \subset \mathcal{R}$ and $B = A^{(k_0)}$, the corresponding subset of $\mathbb{Z}_+$. Thus by (1.12)

$$|B \cdot B| < N^{\frac{b-1}{\ell}} |B|. \tag{1.13}$$

Apply Proposition 2 to the set $B$, $|B| \equiv N_0$, $K = N^{\frac{b-1}{\ell}}$ with $\tau, \gamma$ specified later.

Hence from (1.11)

$$\lambda_q(A) \le \lambda_q(B) < N^{\frac{b-1}{\ell}\Lambda} N_0^\gamma < N^{\frac{b-1}{\ell}\Lambda + b\gamma} \tag{1.14}$$

Taking $q = 2h$, (1.10) and (1.14) imply

$$|hA| > N^{(1-2\frac{b-1}{\ell}\Lambda - 2b\gamma)h} \tag{1.15}$$

Take $h = 2b < k, \gamma = \frac{1}{100b}$. Recall that $\Lambda = \Lambda(\gamma, q)$, hence $\Lambda = \Lambda(b)$. Take $\ell = 100\, b\, \Lambda(b)$, so that $k = 2^\ell \equiv k(b)$. Inequality (1.15) then clearly implies that

$$|kA| > N^b$$

This proves the Theorem.

Returning to Proposition 2, it will suffice to prove the following weaker version

**Proposition 2'.** *Given $\gamma > 0, \tau > 0$ and $q > 2$, and $A$ as in Proposition 2, there is a subset $A' \subset A$ satisfying*

$$|A'| > N^{1-\tau} \tag{1.16}$$

$$\lambda_q(A') < K^\Lambda N^\gamma, \tag{1.17}$$

*where $\Lambda = \Lambda(\tau, \gamma, q)$.*

**Proof of Proposition 2 assuming Proposition 2'.**

Denoting $\chi$ the indicator function, one has obviously

$$\sum_{z \in \frac{A}{A'}} \chi_{zA'} \ge |A'|\, \chi_A \tag{1.18}$$

Let $A'$ be the subset obtained in Proposition 2'. Then (1.18) is easily seen to imply

$$\begin{aligned} |A'|\, \lambda_q(A) &\le \sum_{z \in \frac{A}{A'}} \lambda_q(zA') \\ &= |\frac{A}{A'}|\, \lambda_q(A') \\ &\le |\frac{A}{A}|\, K^\Lambda N^\gamma. \end{aligned} \tag{1.19}$$

If $\mathcal{A} \subset \mathcal{R}$ is the set introduced before, application of Ruzsa's inequality on sum-difference sets gives

$$|\frac{A}{A}| = |\mathcal{A} - \mathcal{A}| \le K^2\,|\mathcal{A}| = K^2 N. \tag{1.20}$$

Thus, by (1.16), (1.19) and (1.20), we have

$$\lambda_q(A) \le K^{\Lambda+2} N^{\tau+\gamma}, \tag{1.21}$$

where $\Lambda = \Lambda(\tau, \gamma, q)$. Replacing $\gamma$ by $\frac{\gamma}{2}$ and $\tau = \frac{\gamma}{2}$, (1.11) follows.

The remainder of the paper is the proof of Proposition 2' which will be rather tedious although elementary. The key statement is Proposition 3 below. This more technical result involves "graphs". We were unable to carry out our analysis by passing simply to subsets. As will be clear later on, the use of graphs allows indeed more flexibility in various constructions involving certain "regularizations". Trying to achieve them using subsets of $A$ is less economical and did not seem to provide us with the desired conclusions. These considerations are particularly relevant to Lemma 3.2 below (which is the base of the multi-scale analysis)and the difficulties encountered with its proof.

## Section 2. Reduction to Proposition 3 and preliminary estimates

Given subsets $A_1, A_2$ in an Abelian group and a symmetric graph $\mathcal{G} \subset A_1 \times A_2$, define

$$A_1 \underset{\mathcal{G}}{+} A_2 = \{x_1 + x_2 | (x_1, x_2) \in \mathcal{G}\}.$$

For technical reason, we will prove a more general (and complicated) version of Proposition 2 (we consider subsets of $\mathcal{R}$, hence work in the additive setting).

**Proposition 3.** *Given $\tau, \gamma > 0$ and $q$, there is a constant $\Lambda = \Lambda(\tau, \gamma, q)$ such that the following holds.*

*Let $\mathcal{P}_0 \subset \mathcal{P}$ be a set of primes, and let $\mathcal{R}_0 = \prod_{p \in \mathcal{P}_0} \mathbb{Z}_{\ge 0}$. Let $A_1,\ A_2 \subset \mathcal{R}_0$ be finite with $|A_i| = N_i$ and let*

$$N = N_1 N_2.$$

*If $\mathcal{G} \subset A_1 \times A_2$ with*

$$|\mathcal{G}| > \delta N,$$

*then there is $\mathcal{G}' \subset \mathcal{G}$ satisfying*

$$|\mathcal{G}'| > \delta^{\Lambda \log\log N} N^{1-\tau} \tag{2.1}$$

*and, moreover, for all $x \in \mathcal{R}_0$, the set $\mathcal{G}'(x) = \{x' \mid (x, x') \in \mathcal{G}',\ or\ (x', x) \in \mathcal{G}'\}$ has the following property:*

*If $(F_\alpha)_{\alpha \in \mathcal{G}'(x)}$ are arbitrary trigonometric polynomials such that $(n, p) = 1$ for all $n \in$ supp $\widehat{F}_\alpha$ and for all $p \in \mathcal{P}_0$, then*

$$\Big\| \sum_\alpha F_\alpha \Big( \prod_{\mathcal{P}_0} p^{\alpha_p} \theta \Big) \Big\|_q \leq K(\mathcal{G})^\Lambda N^\gamma \Big( \sum_\alpha \|F_\alpha\|_q^2 \Big)^{1/2} \tag{2.2}$$

*holds, where we denote*

$$K(\mathcal{G}) = \frac{|A_1 \underset{\mathcal{G}}{+} A_2|}{\sqrt{N_1 N_2}}.$$

*Remark.* If $A_1 = A_2 = A$ and $\mathcal{G} = A \times A$, then $K(\mathcal{G})$ is the doubling constant of $A$.

We will use the following

**Notation.** $M \sim N$ means that there are constants $c$ and $d$ such that $dN < M < cN$.

**Proposition 3 implies Proposition 2'.** Take $A_1 = A_2 = \mathcal{A} \subset \mathcal{R}$, $\mathcal{P}_0 = \mathcal{P}$, and the full graph $\mathcal{G} = A_1 \times A_2$. (Note that $K(\mathcal{G}) = K$, the doubling constant in Proposition 2.) Since $\mathcal{G}' \subset \mathcal{A} \times \mathcal{A}$ satisfies, by (2.1)

$$|\mathcal{G}'| > N^{2-3\tau},$$

there is $x \in \mathcal{A}$ such that $|\mathcal{G}'(x)| > N^{1-3\tau}$. Let $A' \subset A$ be the set corresponding to $\mathcal{G}'(x)$. Thus $|A'| > N^{1-3\tau}$ and, taking $F_\alpha(\theta) = c_\alpha e^{2\pi i \theta}$, $c_\alpha \in \mathbb{R}$, statement (2.2) clearly implies (1.17)

The proof of Proposition 3 will proceed in several stages. In this process, we consider pairs of functions

$$\phi(N, \delta, K), \psi(N, \delta, K)$$

such that under the assumptions of Proposition 3, there is $\mathcal{G}' \subset \mathcal{G}$ satisfying

$$|\mathcal{G}'| > \phi(N, \delta, K(\mathcal{G})) \qquad (N = N_1 N_2) \tag{2.3}$$

and (2.2) holds in the form

$$\Big\| \sum_\alpha F_\alpha \Big( \prod p^{\alpha_p} \theta \Big) \Big\|_q \leq \psi(N, \delta, K(\mathcal{G})) \Big( \sum_\alpha \|F_\alpha\|_q^2 \Big)^{1/2}. \tag{2.4}$$

(We assume $q$ fixed.) We call such a pair of functions *admissible*.

The strategy then consists in getting better and better bounds on the functions $\phi(N,\delta,K),\psi(N,\delta,K)$ and eventually prove Proposition 3. Let us specify a first pair of admissible functions $\phi,\psi$. (see (2.13), (2.14))

Assume $N_1\geq N_2$.

Obviously $|A_1\underset{\mathcal{G}}{+}A_2|\geq\frac{|\mathcal{G}|}{N_2}\geq\delta N_1$, hence

$$\delta N_1<K(\mathcal{G})(N_1N_2)^{1/2}.$$

Namely,

$$N_2>\left(\frac{\delta}{K(\mathcal{G})}\right)^2N_1. \tag{2.5}$$

We may assume that $\mathcal{G}\subset A\times A$ is symmetric, where $A=A_1\cup A_2$, with $|A|\sim N_1$. Let

$$K=K(\mathcal{G}),$$

we have thus

$$\begin{aligned}|\mathcal{G}|&>\delta N_1N_2>\frac{\delta^3}{K^2}N_1^2=\delta_1N_1^2\\ |A\underset{\mathcal{G}}{+}A|&=|A_1\underset{\mathcal{G}}{+}A_2|<KN_1,\end{aligned}$$

where

$$\delta_1=\frac{\delta^3}{K^2}.$$

Applying Gower's version of the Balog-Szemeredi theorem (with powerlike estimate), see [Go], we may find a subset $A'\subset A$ satisfying

$$|A'|>\delta'N_1 \tag{2.6}$$

$$|A'-A'|<K'N_1 \tag{2.7}$$

$$|\mathcal{G}\cap(A'\times A')|>\delta'N_1^2 \tag{2.8}$$

where

$$\delta'>\left(\frac{\delta_1}{K}\right)^C>\left(\frac{\delta}{K}\right)^C,$$

$$K'<\left(\frac{K}{\delta}\right)^C \tag{2.9}$$

(here and in the sequel, notation $C$ as well as $c$ may refer to different constants).

By (2.6), (2.7) and Freiman's lemma, the dimension of the vector space spanned by $A'$ is less than $\frac{K'}{\delta'}$. Thus there is a subset $I \subset \mathcal{P}_0$ such that

$$|I| < \frac{K'}{\delta'} \tag{2.10}$$

and the coordinate restriction $\pi_I : \mathcal{R}_0 \to \prod_{p\in I} \mathbb{Z}_{\geq 0}$ is one-to-one when restricted to $A'$.

Let $\mathcal{G}' = \mathcal{G} \cap (A' \times A')$ satisfying by (2.8), (2.9)

$$|\mathcal{G}'| > \left(\frac{\delta}{K}\right)^C N_1 N_2. \tag{2.11}$$

Fix $x \in \mathcal{R}_0$ and consider $\mathcal{G}'(x) \subset A'$ and trigonometric polynomials $F_\alpha, \alpha \in \mathcal{G}'(x)$ as in (2.2). Thus $(n,p) = 1$, for all $n \in \operatorname{supp} \widehat{F_\alpha}$, and for all $p \in \mathcal{P}_0$. It follows from the preceding that $\alpha \in \mathcal{G}'(x)$ is uniquely determined by $\alpha' = \pi_I(\alpha)$. Therefore clearly

$$F_\alpha\Big(\prod_{p\in\mathcal{P}_0} p^{\alpha_p}\theta\Big) = F'_{\alpha'}\Big(\prod_{p\in I} p^{\alpha_p}\theta\Big)$$

where $(n,p) = 1$ for $n \in \operatorname{supp} \widehat{F'_{\alpha'}}$, and $p \in I$.

Thus Proposition 1 and (2.10) imply

$$\begin{aligned}
\Big\| \sum_{\alpha\in\mathcal{G}'(x)} F_\alpha\big(\prod p^{\alpha_p}\theta\big)\Big\|_q &= \Big\|\sum_\alpha F'_{\alpha'}\Big(\prod_{p\in I} p^{\alpha_p}\theta\Big)\Big\|_q \\
&\leq (Cq)^{|I|}\Big(\sum_\alpha \|F'_{\alpha'}\|_q^2\Big)^{1/2} \\
&\leq (Cq)^{\frac{K'}{\delta'}}\Big(\sum_\alpha \|F_\alpha\|_q^2\Big)^{1/2}.
\end{aligned} \tag{2.12}$$

Hence, (2,11), (2.12), and (2.9) provide the following pair of admissible functions

$$\phi(N,\delta,K) = \left(\frac{\delta}{K}\right)^C N \tag{2.13}$$

$$\psi(N,\delta,K) = \exp\left(\log q \cdot \left(\frac{K}{\delta}\right)^C\right) \tag{2.14}$$

for some constant $C$.

Again the dependence of $\psi$ on $K$ is very poor, since it is a useless bound unless $K \ll \log N$.

The aim of what follows is to improve this dependence of $\psi$ on $K$.

## Section 3. Proof of Proposition 3, Part I: The Factorization

The next statement is a recipe to convert pairs of admissible functions $\phi, \psi$. We will always assume

$\phi, \psi$ are increasing in $N$

$\phi$ is increasing in $\delta$, decreasing in $K$

$\psi$ increases in $K$

and

$$\phi(N, \delta, K) \leq \frac{N}{M} \phi(M, \delta, K) \text{ for } M \leq N. \tag{3.1}$$

**Lemma 3.2.** *Let $\phi, \psi$ be admissible. Define*

$$\tilde{\phi}(N, \delta, K) = \min \phi(N', \delta', K') \cdot \phi(N'', \delta'', K'') \tag{3.3}$$

$$\tilde{\psi}(N, \delta, K) = Cq \max \psi(N', \delta', K') \cdot \psi(N'', \delta'', K'') \tag{3.4}$$

*where in (3.3), (3.4) the range of $N', N'', \delta', \delta'', K', K''$ are as follows*

$$N \geq N'N'' > N \left( \frac{\delta}{\log K} \right)^{40} \tag{3.5}$$

$$N' + N'' < \left( \frac{K}{\delta} \right)^{20} N^{1/2} \tag{3.6}$$

$$\delta' \cdot \delta'' > \left( \log \frac{K}{\delta} \right)^{-6} \delta \tag{3.7}$$

$$K' \cdot K'' < \delta^{-6} (\log K)^{20} K. \tag{3.8}$$

*Then $\tilde{\phi}, \tilde{\psi}$ are also admissible.*

This lemma is an essential ingredient in the proof of Proposition 3. In its proof, the role of graphs will become apparent.

Under the assumption of Proposition 3, we have $\mathcal{G} \subset A_1 \times A_2$, for $A_i \subset \mathcal{R}_0 = \prod_{p\in\mathcal{P}_0} \mathbb{Z}_{\geq 0}$, with $|A_i| = N_i$, and

$$|\mathcal{G}| > \delta N,$$

$$|A_1 \underset{\mathcal{G}}{+} A_2| < K(\mathcal{G})\sqrt{N},$$

where $N = N_1 N_2$.

The proof of Lemma 3.2 is in seven steps.

**Step 1.** *For $i = 1, 2$, we reduce $A_i$ to $A_i'$, with $|A_i'| = N_i'$ such that for any $B_i \subset A_i'$,*

$$|\mathcal{G} \cap (B_1 \times A_2')| > \frac{\delta}{4}|B_1|N_2' \tag{3.9}$$

$$|\mathcal{G} \cap (A_1' \times B_2)| > \frac{\delta}{4}|B_2|N_1' \tag{3.10}$$

*and*

$$N_i' > \frac{3\delta}{4}N_i. \tag{3.11}$$

*Moreover, the property*

$$(\mathcal{G} \cap (A_1' \times A_2')^c| \leq \frac{\delta}{4}|(A_1 \times A_2)\backslash(A_1' \times A_2')| \tag{3.12}$$

*will hold.*

Thus (3.12) implies (3.11), because

$$N_1'N_2' \geq |\mathcal{G} \cap (A_1' \times A_2')| > \delta N_1 N_2 - \frac{\delta}{4}N_1 N_2 = \frac{3\delta}{4}N_1 N_2.$$

The construction is straightforward. Assume $A_1' \times A_2'$ fails (3.9). Thus $|\mathcal{G} \cap (B_1 \times A_2')| \leq \frac{\delta}{4}|B_1|\ |A_2'|$ for some $B_1 \subset A_1'$. Define $A_1'' = A_1' \backslash B_1$, then

$$\begin{aligned}|\mathcal{G} \cap (A_1'' \times A_2')^c| &= |\mathcal{G} \cap (A_1' \times A_2')^c| + |\mathcal{G} \cap (B_1 \times A_2')| \\ &\leq \frac{\delta}{4}|(A_1 \times A_2)\backslash(A_1' \times A_2')| + \frac{\delta}{4}|B_1|\ |A_2'| \\ &= \frac{\delta}{4}|(A_1 \times A_2)\backslash(A_1'' \times A_2')|\end{aligned}$$

and (3.12) remains valid.

Continuing removing the bad set $B_i$, (3.12) ensures that the remaining set is still big enough, and the process gives the desired result.

**Step 2.** *We decompose $\mathcal{P}_0 \subset \mathcal{P}$ in disjoint sets $\mathcal{P}_0 = \mathcal{P}_1 \cup \mathcal{P}_2$.*

The choice of this decomposition will only matter for condition (3.6).

We proceed as follows

Enumerate $\mathcal{P}_0 = \{p_1 < p_2 < \cdots < p_t\}$ which we identify with $\{1, \ldots, t\}$,

For $t' \leq t$, consider the decreasing functions $(i = 1, 2)$

$$n_i(t') = \max_{(x_1, \ldots, x_{t'}) \in \mathbb{Z}^{t'}} |A_i(x_1, \ldots, x_{t'})|,$$

where $A_i(x_1, \ldots, x_{t'}) = \{(x_{t'+1}, \ldots, x_t) \mid (x_1, \ldots, x_t) \in A_i\}$.

We take $t'$ such that

$$\begin{cases} n_1(t') + n_2(t') \geq (N_1 N_2)^{1/4} \\ n_1(t'+1) + n_2(t'+1) \leq (N_1 N_2)^{1/4}. \end{cases} \tag{3.13}$$

We assume $n_1(t') \geq n_2(t')$, thus

$$n_1(t') \geq \frac{1}{2} N^{1/4}. \tag{3.14}$$

Decompose then $\mathcal{P}_0 = \mathcal{P}_1 \cup \mathcal{P}_2$ where $\mathcal{P}_1 = \{p_1, \ldots, p_{t'}\}$.

**Step 3.** *Let $\mathcal{R}_i = \prod_{p \in \mathcal{P}_i} \mathbb{Z}_{\geq 0}$ with $\mathcal{R}_0 = \mathcal{R}_1 \times \mathcal{R}_2$ corresponding to the decomposition in Step 2, and let $\pi_1 : \mathcal{R}_0 \to \mathcal{R}_1$ be the projection to the first $t'$ coordinates. Denote*

$$\bar{x} = (x_1, \cdots, x_{t'}).$$

*We construct a set $\bar{\bar{A}}_2 \subset A_2'$ such that for all $\bar{x} \in \pi_1(\bar{\bar{A}}_2)$, we have $|\bar{\bar{A}}_2(\bar{x})| \sim m_2 > c\delta^5 K^{-2} N^{1/4}$ and $M_2 \equiv |\pi_1(\bar{\bar{A}}_2)| \sim \frac{|\bar{\bar{A}}_2|}{m_2} < C\delta^{-5} K^2 \frac{N_2}{N^{1/4}}$, and $\bar{\bar{N}}_2 \equiv |\bar{\bar{A}}_2| > c\frac{\delta^3}{\log \frac{K}{\delta}} N_2$.*

Choose $\bar{x} \in \pi_1(A_1')$ such that

$$|A_1'(\bar{x})| = n_1(t'). \tag{3.15}$$

It follows from (3.9) that

$$|\mathcal{G} \cap [(\{\bar{x}\} \times A_1'(\bar{x})) \times A_2']| > \frac{\delta}{4} n_1(t') N_2'$$

and hence there is a subset $A_2'' \subset A_2'$ such that by Fact 1 below,

$$N_2'' = |A_2''| > \frac{\delta}{8} N_2' \tag{3.16}$$

and for $z \in A_2''$

$$|\mathcal{G} \cap [(\{\bar{x}\} \times A_1'(\bar{x})) \times \{z\}]) > \frac{\delta}{8} n_1(t'). \tag{3.17}$$

**Fact 1.** *Let $|E| \leq e$ and $|F| \leq f$. If $|\mathcal{G} \cap (E \times F)| > \alpha e f$, then there exists $F' \subset F$ with $|F'| > \frac{\alpha}{2} f$, such that for any $z \in F'$, $|\mathcal{G} \cap (E \times \{z\})| > \frac{\alpha}{2} e$.*

From (2.5) and (3.17), we get clearly

$$\begin{aligned} \frac{K^2}{\delta} N_2 \geq K\sqrt{N_1 N_2} &= |A_1 \underset{\mathcal{G}}{+} A_2| \\ &\geq \left|\left(\{\bar{x}\} \times A_1'(\bar{x})\right) \underset{\mathcal{G}}{+} A_2''\right| > \frac{\delta}{8} |\pi_1(A_2'')| \cdot n_1(t'). \end{aligned} \tag{3.18}$$

Let $\bar{A}_2 \subset A_2''$ such that the fibers over $\bar{x}$ have size at least $\frac{\delta^5 n_1(t')}{10^4 K^2}$, thus

$$\bar{A}_2 = \bigcup_{|A_2''(\bar{x})| > 10^{-4}\delta^5 K^{-2} n_1(t')} \left(\{\bar{x}\} \times A_2''(\bar{x})\right).$$

It follows from (3.18) that

$$|A_2'' \backslash \bar{A}_2| \leq |\pi_1(A_2'')| 10^{-4} \delta^5 K^{-2} n_1(t') < \delta^3 10^{-3} N_2 < \frac{\delta}{10} N_2'' \tag{3.19}$$

The last inequality is by (3.11) and (3.16).

Since by (3.10)

$$|\mathcal{G} \cap (A_1' \times A_2'')| > \frac{\delta}{4} N_1' N_2'',$$

it follows from (3.19) that

$$|\mathcal{G} \cap (A_1' \times \bar{A}_2)| > \frac{\delta}{4} N_1' N_2'' - \frac{\delta}{10} N_1' N_2'' > \frac{\delta}{10} N_1' N_2''. \tag{3.20}$$

Since $|A_2''(\bar{x})| \leq n_2(t') \leq n_1(t')$, we may specify $m_2$ and $\bar{\bar{A}}_2$ as follows:

$$10^{-4} \delta^5 K^{-2} n_1(t') < m_2 < n_1(t'), \tag{3.21}$$

and

$$A_2' \supset A_2'' \supset \bar{A}_2 \supset \bar{\bar{A}}_2 = \bigcup_{|A_2''(\bar{x})| \sim m_2} \left(\{\bar{x}\} \times A_2''(\bar{x})\right) \tag{3.22}$$

such that

$$|\mathcal{G} \cap (A_1' \times \bar{\bar{A}}_2)| > c \frac{\delta}{\log \frac{K}{\delta}} N_1' N_2''. \tag{3.23}$$

Thus $\bar{\bar{N}}_2 = |\bar{\bar{A}}_2|$ satisfies

$$\bar{\bar{N}}_2 > c\frac{\delta}{\log\frac{K}{\delta}} N_2'' > c\frac{\delta^3}{\log\frac{K}{\delta}} N_2. \tag{3.24}$$

The set $\bar{\bar{A}}_2$ has a 'regular' structure with respect to the decomposition $\mathcal{R}_0 = \mathcal{R}_1 \times \mathcal{R}_2$ in the sense that for all $\bar{x} \in \pi_1(\bar{\bar{A}}_2)$, with $|\bar{\bar{A}}_2(\bar{x})| \sim m_2$. In particular, denoting $M_2 = |\pi_1(\bar{\bar{A}}_2)|$, we have

$$\bar{\bar{N}}_2 \sim M_2 \cdot m_2. \tag{3.24'}$$

By (3.21) and (3.14)

$$m_2 > c\delta^5 K^{-2} N^{1/4},$$

and

$$M_2 < C\delta^{-5} K^2 \frac{N_2}{N^{1/4}}. \tag{3.25}$$

**Step 4.** *Regularization of $A_1'$. We construct a set $\bar{\bar{A}}_1 \subset A_1'$ such that for any $\bar{x} \in \pi_1(\bar{\bar{A}}_1)$, we have $|\bar{\bar{A}}_1(\bar{x})| \sim m_1 > c\delta^{10} K^{-5} N^{1/4}$, $M_1 \equiv |\pi_1(\bar{\bar{A}}_1)| \sim \frac{|\bar{\bar{A}}_1|}{m_1} < C\delta^{-10} K^5 \frac{N_1}{N^{1/4}}$, $\bar{\bar{N}}_1 = |\bar{\bar{A}}_1| > c\frac{\delta^2}{(\log\frac{K}{\delta})^2} N_1$, and $|\mathcal{G} \cap (\bar{\bar{A}}_1 \times \bar{\bar{A}}_2)| > c\frac{\delta}{(\log\frac{K}{\delta})^2} \bar{\bar{N}}_1 \bar{\bar{N}}_2$.*

**Claim.** *Let $\tilde{A}_1 \subset A_1'$. If*

$$|\mathcal{G} \cap (\tilde{A}_1 \times \bar{\bar{A}}_2)| \sim |\mathcal{G} \cap (A_1' \times \bar{\bar{A}}_2)|, \tag{3.26}$$

*then*

$$m \equiv \max_{\bar{x} \in \pi_1(\tilde{A}_1)} |\tilde{A}_1(\bar{x})| > c\frac{\delta^4}{(\log\frac{K}{\delta})^2} K^{-2} m_2. \tag{3.27}$$

**Proof of Claim.** From (3.26), (3.23) and the regular structure of $\bar{\bar{A}}_2$, there is $\bar{x} \in \pi_1(\bar{\bar{A}}_2)$ such that

$$|\mathcal{G} \cap \big(\tilde{A}_1 \times \big(\{\bar{x}\} \times \bar{\bar{A}}_2(\bar{x})\big)\big)| > c\frac{\delta}{\log\frac{K}{\delta}} N_1' m_2.$$

Hence by Fact 1, there is a subset $A_1'' \subset \tilde{A}_1$ satisfying

$$|A_1''| > c\frac{\delta}{\log\frac{K}{\delta}} N_1' \tag{3.28}$$

and for any $z \in A_1''$

$$|\mathcal{G} \cap (\{z\} \times (\{\bar{x}\} \times \bar{\bar{A}}_2(\bar{x})))| > c\frac{\delta}{\log \frac{K}{\delta}} m_2.$$

As in Step 3, (3.18), write

$$\begin{aligned}\frac{K^2}{\delta} N_1 \geq K\sqrt{N_1 N_2} \geq |A_1 \underset{\mathcal{G}}{+} A_2| \geq |A_1'' \underset{\mathcal{G}}{+} \big(\{\bar{x}\} \times \bar{\bar{A}}_2(\bar{x})\big)| \\ > c|\pi_1(A_1'')| \frac{\delta}{\log \frac{K}{\delta}} m_2 \\ > c\frac{|A_1''|}{m} \ \frac{\delta}{\log \frac{K}{\delta}} m_2 \\ > c\frac{\delta^3}{(\log \frac{K}{\delta})^2} \ \frac{m_2}{m} N_1.\end{aligned}$$

The last two inequalities are by the definition of $m$ in (3.27) and (3.28), (3.11). Hence

$$m > c\frac{\delta^4}{(\log \frac{K}{\delta})^2} K^{-2} m_2. \tag{3.29}$$

Clearly, the bound in (3.27) is smaller than $\delta^5 K^{-3} m_2$. Therefore, in (3.23) we may replace $A_1'$ by $\bar{A}_1$ defined as follows.

$$A_1' \supset \bar{A}_1 = \bigcup_{|A_1'(\bar{x})| > \delta^5 K^{-3} m_2} \big(\{\bar{x}\} \times A_1'(\bar{x})\big).$$

Thus

$$|\mathcal{G} \cap (\bar{A}_1 \times \bar{\bar{A}}_2)| > c\frac{\delta}{\log \frac{K}{\delta}} N_1' N_2''.$$

Recalling (3.21), for $\bar{x} \in \pi_1(\bar{A}_1)$

$$\delta^5 K^{-3} m_2 < |\bar{A}_1(\bar{x})| \leq n_1(t') < C\delta^{-5} K^2 m_2.$$

Keeping (3.23) and (3.26) in mind, we may thus again specify

$$\delta^5 K^{-3} m_2 < m_1 < C\delta^{-5} K^2 m_2 \tag{3.30}$$

such that the regular set $\bar{\bar{A}}_1$ defined as

$$A_1' \supset \bar{A}_1 \supset \bar{\bar{A}}_1 = \bigcup_{|\bar{A}_1(\bar{x})| \sim m_1} \big(\{\bar{x}\} \times \bar{A}_1(\bar{x})\big) \tag{3.31}$$

will satisfy

$$|\mathcal{G} \cap (\bar{\bar{A}}_1 \times \bar{\bar{A}}_2)| > c \frac{\delta}{(\log \frac{K}{\delta})^2} N_1' N_2''. \tag{3.32}$$

Denoting $\bar{\bar{N}}_1 = |\bar{\bar{A}}_1|, M_1 = |\pi_1(\bar{\bar{A}}_1)|$, we have $\bar{\bar{N}}_1 \sim M_1 m_1$. On the other hand, (3.32), (3.11) and the fact that $\bar{\bar{A}}_i \subset A_i''$ give

$$\bar{\bar{N}}_1 > c \frac{\delta^2}{(\log \frac{K}{\delta})^2} N_1 \tag{3.33}$$

and

$$|\mathcal{G} \cap (\bar{\bar{A}}_1 \times \bar{\bar{A}}_2)| > c \frac{\delta}{(\log \frac{K}{\delta})^2} \bar{\bar{N}}_1 \bar{\bar{N}}_2. \tag{3.34}$$

It follows from (3.25) and (3.30) that

$$m_1 > c\delta^{10} K^{-5} N^{1/4},$$

$$M_1 < C\delta^{-10} K^5 \frac{N_1}{N^{1/4}}. \tag{3.35}$$

Thus at this stage we have regularized both $A_1, A_2$ with respect to the decomposition $\mathcal{P}_0 = \mathcal{P}_1 \cup \mathcal{P}_2$.

*For simplicity, we re-denote $\bar{\bar{A}}_1, \bar{\bar{A}}_2$ by $A_1, A_2$ whose cardinalities $\bar{\bar{N}}_i \sim m_i M_i$ satisfying (3.24) and (3.33).*

**Step 5.** *Regularization of the graph. We construct $\mathcal{G}_{1,0} \subset \pi_1(A_1) \times \pi_1(A_2) \subset \mathcal{R}_1 \times \mathcal{R}_1$ with $|\mathcal{G}_{1,0}| > \delta_0 M_1 M_2$, such that $\forall (\bar{x}_1, \bar{x}_2) \in \mathcal{G}_{1,0}$, we have $|A_1(\bar{x}_1) \underset{\mathcal{G}_{\bar{x}_1,\bar{x}_2}}{+} A_2(\bar{x}_2)| \sim L\sqrt{m_1 m_2}$, and $|\mathcal{G}_{\bar{x}_1,\bar{x}_2}| \sim \delta_1 m_1 m_2$, where $\mathcal{G}_{\bar{x}_1,\bar{x}_2}$ is the fiber over $(\bar{x}_1, \bar{x}_2)$, and $\delta_0, \delta_1$ and $L$ satisfy (3.45), (3.38) and (3.43) respectively.*

For $\bar{x}_1, \bar{x}_2 \in \mathcal{R}_1$, let $\mathcal{G}_{\bar{x}_1,\bar{x}_2}$ be the fiber over $(\bar{x}_1, \bar{x}_2)$,

$$\mathcal{G}_{\bar{x}_1,\bar{x}_2} = \{(\bar{y}_1, \bar{y}_2) \in A_1(\bar{x}_1) \times A_2(\bar{x}_2) | \big((\bar{x}_1, \bar{y}_1), (\bar{x}_2, \bar{y}_2)\big) \in \mathcal{G}\} \subset \mathcal{R}_2 \times \mathcal{R}_2.$$

It follows from (3.34) that we may restrict $\mathcal{G}$ to $\mathcal{G}_1 \times (\mathcal{R}_2 \times \mathcal{R}_2)$, where

$$\mathcal{G}_1 = \{(\bar{x}_1, \bar{x}_2) \in \pi_1(A_1) \times \pi_1(A_2) | \; |\mathcal{G}_{\bar{x}_1,\bar{x}_2}| > c \frac{\delta}{(\log \frac{K}{\delta})^2} m_1 m_2\}. \tag{3.36}$$

Thus

$$c\, m_1 m_2 \geq |\mathcal{G}_{\bar{x}_1,\bar{x}_2}| > c \frac{\delta}{(\log \frac{K}{\delta})^2} m_1 m_2, \text{ for } (\bar{x}_1, \bar{x}_2) \in \mathcal{G}_1$$

and by (3.34)

$$\sum_{(\bar{x}_1,\bar{x}_2)\in\mathcal{G}_1} |\mathcal{G}_{\bar{x}_1,\bar{x}_2}| > c \frac{\delta}{(\log \frac{K}{\delta})^2} \bar{\bar{N}}_1 \bar{\bar{N}}_2. \tag{3.37}$$

We may thus specify $\delta_1$,

$$1 > \delta_1 > c \frac{\delta}{(\log \frac{K}{\delta})^2} \tag{3.38}$$

such that if

$$\mathcal{G}_1' = \{(\bar{x}_1, \bar{x}_2) \in \mathcal{G}_1 | \; |\mathcal{G}_{\bar{x}_1,\bar{x}_2}| \sim \delta_1 m_1 m_2\}, \tag{3.39}$$

then we have

$$\sum_{(\bar{x}_1,\bar{x}_2)\in\mathcal{G}_1'} |\mathcal{G}_{\bar{x}_1,\bar{x}_2}| > c \frac{\delta}{(\log \frac{K}{\delta})^3} \bar{\bar{N}}_1 \bar{\bar{N}}_2. \tag{3.40}$$

Hence

$$|\mathcal{G}_1'| > c \frac{\delta}{\delta_1 (\log \frac{K}{\delta})^3} M_1 M_2, \tag{3.41}$$

which is bigger than $\frac{\delta}{(\log \frac{K}{\delta})^3} M_1 M_2$.

By further restriction of $\mathcal{G}_1'$, we will also make a specification on the size of the sumset of $\mathcal{G}_{\bar{x}_1,\bar{x}_2}$.

For $(\bar{x}_1, \bar{x}_2) \in \mathcal{G}_1'$, let $K(\mathcal{G}_{\bar{x}_1,\bar{x}_2})$ be the addition constant of $A_1(\bar{x}_1)$ and $A_2(\bar{x}_2)$ along the graph $\mathcal{G}_{\bar{x}_1,\bar{x}_2}$ as defined in Proposition 3. Let $\mathcal{H} \subset \mathcal{G}_1'$, with

$$|\mathcal{H}| \sim |\mathcal{G}_1'| > \frac{\delta}{(\log \frac{K}{\delta})^3} M_1 M_2. \tag{3.42}$$

**Claim.**

$$\min_{(\bar{x}_1,\bar{x}_2)\in\mathcal{H}} K(\mathcal{G}_{\bar{x}_1,\bar{x}_2}) < L_0 \equiv (\log\frac{K}{\delta})^{\frac{9}{2}}\delta^{-\frac{9}{2}}K. \tag{3.43}$$

**Proof.** Assume for all $(\bar{x}_1,\bar{x}_2)\in\mathcal{H}$ that $K(\mathcal{G}_{\bar{x}_1,\bar{x}_2}) > L_0$. Then

$$\begin{aligned} K\sqrt{N_1N_2} \ge |A_1 \underset{\mathcal{G}}{+} A_2| &> \min_{(\bar{x}_1,\bar{x}_2)\in\mathcal{H}}\{|A_1(\bar{x}_1) \underset{\mathcal{G}_{\bar{x}_1,\bar{x}_2}}{+} A_2(\bar{x}_2)|\}|\pi_1(A_1) \underset{\mathcal{H}}{+} \pi_1(A_2)| \\ &\ge L_0\sqrt{m_1m_2}\ \frac{|\mathcal{H}|}{\sqrt{M_1M_2}} > L_0\frac{c\,\delta}{(\log\frac{K}{\delta})^3}\,(\bar{\bar{N}}_1\bar{\bar{N}}_2)^{1/2} \\ &> c\,\delta^{-1}\sqrt{N_1N_2}K, \end{aligned}$$

which is a contradiction. (The last inequality is by (3.24), (3.33) and (3.43).)

Hence, we may reduce $\mathcal{G}_1'$ to $\mathcal{G}_1''\subset\mathcal{G}_1'$, with $|\mathcal{G}_1''|\sim|\mathcal{G}_1'|$ such that

$$|A_1(\bar{x}_1)\underset{\mathcal{G}_{\bar{x}_1,\bar{x}_2}}{+}A_2(\bar{x}_2)| < L_0\sqrt{m_1m_2} \text{ for } (\bar{x}_1,\bar{x}_2)\in\mathcal{G}_1''.$$

Therefore there is $\mathcal{G}_{1,0}\subset\mathcal{G}_1''$ and $L<L_0$ (see (3.43))

$$|\mathcal{G}_{1,0}| > \frac{|\mathcal{G}_1''|}{\log\frac{K}{\delta}} > \delta_0M_1M_2, \tag{3.44}$$

where, by (3.41)

$$\delta_0 > c\,\frac{\delta}{\delta_1(\log\frac{K}{\delta})^4} \tag{3.45}$$

and

$$|A_1(\bar{x}_1)\underset{\mathcal{G}_{\bar{x}_1,\bar{x}_2}}{+}A_2(\bar{x}_2)| \sim L\sqrt{m_1m_2} \tag{3.46}$$

for $(\bar{x}_1,\bar{x}_2)\in\mathcal{G}_{1,0}$.

Since

$$\begin{aligned} K\sqrt{N_1N_2} &\ge |\pi_1(A_1)\underset{\mathcal{G}_{1,0}}{+}\pi_1(A_2)||A_1(\bar{x}_1)\underset{\mathcal{G}_{\bar{x}_1,\bar{x}_2}}{+}A_2(\bar{x}_2)| \\ &\ge |\pi_1(A_1)\underset{\mathcal{G}_{1,0}}{+}\pi_1(A_2)|\cdot L\sqrt{m_1m_2} \\ &= K(\mathcal{G}_{1,0})L\sqrt{\bar{\bar{N}}_1\bar{\bar{N}}_2}, \end{aligned}$$

we have

$$K(\mathcal{G}_{1,0})\cdot L<\delta^{-\frac{5}{2}}(\log\frac{K}{\delta})^{\frac{3}{2}}K<\delta^{-3}(\log K)^2K. \tag{3.47}$$

In summary, $\mathcal{G}_{1,0}\subset\pi_1(A_1)\times\pi_1(A_2)$ satisfies (3.44), (3.45) and for $(\bar{x}_1,\bar{x}_2)\in\mathcal{G}_{1,0}$, the graph $\mathcal{G}_{\bar{x}_1,\bar{x}_2}\subset A_1(\bar{x}_1)\times A_2(\bar{x}_2)$ satisfies

$$\begin{aligned}&\{(\bar{x}_1,\bar{x}_2)\}\times\mathcal{G}_{\bar{x}_1,\bar{x}_2}\subset\mathcal{G}\\&|\mathcal{G}_{\bar{x}_1,\bar{x}_2}|\sim\delta_1 m_1m_2,\end{aligned} \tag{3.48}$$

where $\delta_1$ is as in (3.38). The addition constants $K(\mathcal{G}_{1,0})$ and $L$ satisfy (3.43) and (3.47).

Denote

$$\mathcal{G}\supset\tilde{\mathcal{G}}=\bigcup_{(\bar{x}_1,\bar{x}_2)\in\mathcal{G}_{1,0}}(\{(\bar{x}_1,\bar{x}_2)\}\times\mathcal{G}_{\bar{x}_1,\bar{x}_2}) \tag{3.49}$$

which satisfies

$$|\tilde{\mathcal{G}}|>c\,\frac{\delta}{(\log\frac{K}{\delta})^4}\,\bar{\bar{N}}_1\bar{\bar{N}}_2 \tag{3.50}$$

where

$$\bar{\bar{N}}_1\cdot\bar{\bar{N}}_2>\frac{\delta^5}{(\log\frac{K}{\delta})^3}N_1N_2. \tag{3.51}$$

**Step 6.** *Moment inequalities*

Let $\tilde{\mathcal{G}}$ be the graph obtained in (3.49). We reduce further $\tilde{\mathcal{G}}$ to a graph $\mathcal{G}'$ to fulfil condition (2.4).

Consider first the graph $\mathcal{G}_{1,0}\subset\pi_1(A_1)\times\pi_1(A_2)\subset\mathcal{R}_1\times\mathcal{R}_1$ and denote $K_0=K(\mathcal{G}_{1,0})$. By (3.44) and since $\phi,\psi$ are admissible, there is $\mathcal{G}'_{1,0}\subset\mathcal{G}_{1,0}$ satisfying

$$|\mathcal{G}'_{1,0}|>\phi(M_1M_2,\delta_0,K_0) \tag{3.52}$$

and

$$\Big\|\sum F_\alpha(\prod_{p\in\mathcal{P}_1}p^{\alpha_p}\theta)\Big\|_q\le\psi(M_1M_2,\delta_0,K_0)\Big(\sum\|F_\alpha\|_q^2\Big)^{1/2} \tag{3.53}$$

whenever $\bar{x}\in\mathcal{R}_1$, $(F_\alpha)_{\alpha\in\mathcal{G}'_{1,0}(\bar{x})}$ trigonometric polynomials satisfying

$$(n,p)=1,\ \text{for any } n\in\operatorname{supp}\widehat{F}_\alpha,\ \text{and for any } p\in\mathcal{P}_1. \tag{*}$$

Next, fix $(\bar{x}_1, \bar{x}_2) \in \mathcal{G}'_{1,0}$ and consider the graph $\mathcal{G}_{\bar{x}_1,\bar{x}_2} \subset A_1(\bar{x}_1) \times A_2(\bar{x}_2) \subset \mathcal{R}_2 \times \mathcal{R}_2$ satisfying (3.48) and (3.46).

Thus there is a subgraph $\mathcal{G}'_{\bar{x}_1,\bar{x}_2} \subset \mathcal{G}_{\bar{x}_1,\bar{x}_2}$ s.t.

$$|\mathcal{G}'_{\bar{x}_1,\bar{x}_2}| > \phi(m_1 m_2, \delta_1, L) \tag{3.54}$$

and

$$\Big\| \sum G_\alpha \Big( \prod_{p \in \mathcal{P}_2} p^{\alpha_p} \theta \Big) \Big\|_q \leq \psi(m_1 m_2, \delta_1, L) \Big( \sum \|G_\alpha\|_q^2 \Big)^{1/2} \tag{3.55}$$

whenever $\bar{y} \in \mathcal{R}_2$ and $(G_\alpha)_{\alpha \in \mathcal{G}'_{\bar{x}_1,\bar{x}_2}(\bar{y})}$ trigonometric polynomials satisfying

$$(n, p) = 1, \text{ for any } n \in \operatorname{supp} \widehat{G}_\alpha, \text{ and for any } p \in \mathcal{P}_2. \tag{$**$}$$

Consider then the subgraph $\mathcal{G}' \subset \tilde{\mathcal{G}} \subset \mathcal{G}$

$$\mathcal{G}' = \bigcup_{(\bar{x}_1,\bar{x}_2) \in \mathcal{G}'_{1,0}} (\{(\bar{x}_1, \bar{x}_2)\} \times \mathcal{G}'_{\bar{x}_1,\bar{x}_2}) \tag{3.56}$$

which satisfies by (3.52) and (3.54)

$$|\mathcal{G}'| > \phi(M_1 M_2, \delta_0, K_0) \cdot \phi(m_1 m_2, \delta_1, L). \tag{3.57}$$

Next, we check the moment inequality.

Fix thus $x = (\bar{x}, \bar{y}) \in \mathcal{R}_1 \times \mathcal{R}_2$ and consider trigonometric polynomials $(G_\alpha)_{\alpha \in \mathcal{G}'(x)}$, where $\mathcal{G}'(x) = \{\alpha \mid (x, \alpha) \in \mathcal{G}' \subset A \times A\}$, such that

$$(n, p) = 1, \text{ for any } n \in \operatorname{supp} \widehat{G}_\alpha, \text{ and for any } p \in \mathcal{P}_0 = \mathcal{P}_1 \cup \mathcal{P}_2.$$

We need to estimate $\| \sum_\alpha G_\alpha (\prod_{p \in \mathcal{P}_0} p^{\alpha_p} \theta) \|_q$ using (3.53) and (3.55). First, by (3.56)

$$\mathcal{G}'(x) = \bigcup_{\beta \in \mathcal{G}'_{1,0}(\bar{x})} \big(\{\beta\} \times \mathcal{G}'_{\bar{x},\beta}(\bar{y})\big) \subset \mathcal{R}_0.$$

Denote for $\beta \in \mathcal{G}'_{1,0}(\bar{x})$

$$F_\beta(\theta) = \sum_{\pi_1(\alpha) = \beta} G_\alpha \Big( \prod_{p \in \mathcal{P}_2} p^{\alpha_p} \theta \Big)$$

which clearly satisfy $(*)$.

Hence, applying consecutively (3.55) and (3.53)

$$\Big\|\sum_\alpha G_\alpha\Big(\prod_{p\in\mathcal{P}_0} p^{\alpha_p}\theta\Big)\Big\|_q = \Big\|\sum_{\beta\in\mathcal{G}'_{1,0}(\bar x)} F_\beta\Big(\prod_{p\in\mathcal{P}_1} p^{\beta_p}\theta\Big)\Big\|_q$$
$$\le \psi(M_1M_2,\delta_0,K_0)\Big(\sum \|F_\beta\|_q^2\Big)^{1/2}$$
$$\le \psi(M_1M_2,\delta_0,K_0).\,\psi(m_1m_2,\delta_1,L)\Big(\sum_\alpha \|G_\alpha\|_q^2\Big)^{1/2}. \tag{3.58}$$

Returning to the statement in Lemma 3.2 and inequalities (3.57) and (3.58), we get in both (3.3) and (3.4)

$$\begin{cases} N' = M_1M_2 \qquad N'' = m_1m_2\\ \delta' = \delta_0 \qquad \delta'' = \delta_1\\ K' = K_0 \qquad K'' = L\end{cases}$$

Condition (3.5) follows from (3.51) (which is clearly much stronger,) and, restating (3.45) and (3.47)

$$\begin{aligned}&\delta_0\delta_1 > \frac{\delta}{(\log\frac{K}{\delta})^4},\\ &K_0L < \delta^{-3}(\log K)^2K.\end{aligned} \tag{3.59}$$

It remains to consider condition (3.6).

By (3.25) and (3.35)

$$M_1M_2 < C\delta^{-15}K^7N^{1/2} \tag{3.60}$$

but we don't have necessarily the desired bound on $m_1m_2$. To achieve this, we will redefine $\mathcal{G}'_{\bar x_1,\bar x_2}$ by performing one more step in the construction

**Step 7** Recalling Step 2, decompose $\mathcal{P}_2 = \{p_{t'+1},\cdots,p_t\}$ further as

$$\mathcal{P}_2 = \{p_{t'+1}\}\cup\mathcal{P}_3,\ \text{ where } \mathcal{P}_3 = \{p_{t'+2},\cdots,p_t\}.$$

For fixed $(\bar x_1,\bar x_2)\in\mathcal{G}'_{1,0}$, consider the graph $\mathcal{K} = \mathcal{G}_{\bar x_1,\bar x_2}\subset A_1(\bar x_1)\times A_2(\bar x_2)\subset\mathcal{R}_2\times\mathcal{R}_2$ satisfying by (3.46) and (3.48)

$$\begin{aligned}&|A_1(\bar x_1)|\sim m_1,\ \ |A_2(\bar x_2)|\sim m_2\\ &|\mathcal{G}_{\bar x_1,\bar x_2}|\sim\delta_1m_1m_2\\ &K(\mathcal{G}_{\bar x_1,\bar x_2})\sim L.\end{aligned}$$

Repeat then Steps 1 - 5 from previous construction to the graph $\mathcal{K}$ with respect to the decomposition $\mathcal{P}_2 = \{p_{t'+1}\} \cup \mathcal{P}_3$. Thus $\mathcal{K}$ gets replaced by

$$\tilde{\mathcal{K}} = \bigcup_{(z_1,z_2)\in\mathcal{K}_{1,0}} \mathcal{K}_{z_1,z_2} \tag{3.61}$$

where

$$\mathcal{K}_{1,0} \subset (\mathbb{Z}_{\geq 0})^2.$$

Thus $\tilde{\mathcal{K}} \subset \bar{\bar{A}}_1(\bar{x}_1) \times \bar{\bar{A}}_2(\bar{x}_2) \subset A_1(\bar{x}_1) \times A_2(\bar{x}_2)$,

$$\mathcal{K}_{z_1,z_2} \subset \bar{\bar{A}}_1(\bar{x}_1, z_1) \times \bar{\bar{A}}_2(\bar{x}_2, z_2)$$

$$m_i \geq |\bar{\bar{A}}_i(\bar{x}_i)| \equiv \bar{\bar{m}}_i > \frac{\delta_1^3}{(\log\frac{L}{\delta_1})^2} m_i \tag{3.62}$$

$$|\bar{\bar{A}}_i(\bar{x}_i, z_i)| \sim \ell_i \leq |A_i(\bar{x}_i, z_i)| < (N_1N_2)^{1/4} \tag{3.63}$$

$\big($by (3.13)$\big)$

$$|\mathcal{K}_{z_1,z_2}| \sim \delta_3\ell_1\ell_2 \tag{3.64}$$

$$|\mathcal{K}_{1,0}| > \frac{\delta_1}{\delta_3(\log\frac{L}{\delta_1})^4}\ \frac{\bar{\bar{m}}_1\bar{\bar{m}}_2}{\ell_1\ell_2} \tag{3.65}$$

(cf. (3.44), (3.45))

$$K(\mathcal{K}_{z_1,z_2}) < K(\mathcal{K}_{1,0}) \cdot K(\mathcal{K}_{z_1,z_2}) < \delta_1^{-3}(\log L)^2 L \tag{3.66}$$

$\big($cf. (3.47)$\big)$.

(We point out here that $\ell_i, \bar{\bar{m}}_i, \delta_3 > \frac{\delta_1}{(\log\frac{L}{\delta_1})^2}$ do depend on the basepoint $(\bar{x}_1, \bar{x}_2) \in \mathcal{R}_1 \times \mathcal{R}_1$). Starting from (3.61), we carry out Step 6. However, since $\mathcal{K}_{1,0} \subset (\mathbb{Z}_{\geq 0})^2$ (only the prime $p_{t'+1}$ is involved) we may take $\mathcal{K}'_{1,0} = \mathcal{K}_{1,0}$ and replace in (3.53) the factor $\psi(\ )$ by $Cq$ (apply Proposition 1 with $k = 1$). For each $(z_1, z_2) \in \mathcal{K}_{1,0}$, consider again a subgraph $\mathcal{K}'_{z_1,z_2} \subset \mathcal{K}_{z_1,z_2}$ satisfying

$$|\mathcal{K}'_{z_1,z_2}| > \phi\big(\ell_1\ell_2, \delta_3, K(\mathcal{K}_{z_1,z_2})\big) \tag{3.67}$$

and

$$\Big\| \sum G_\alpha\Big(\prod_{p\in\mathcal{P}_3} p^{\alpha_p}\theta\Big)\Big\|_q = \psi\big(\ell_1\ell_2, \delta_3, K(\mathcal{K}_{z_1,z_2})\big)\Big(\sum \|G_\alpha\|_q^2\Big)^{1/2} \tag{3.68}$$

whenever $(G_\alpha)_{\alpha\in\mathcal{K}'_{z_1,z_2}}(\bar y)$ are trigonometric polynomials satisfying

$$(n,p)=1, \text{ for any } n\in \operatorname{supp}\widehat{G}_\alpha, \text{ and for any } p\in\mathcal{P}_3.$$

Redefine then $\mathcal{G}'_{\bar x_1,\bar x_2}\subset\mathcal{G}_{\bar x_1,\bar x_2}$ as

$$\mathcal{G}'_{\bar x_1,\bar x_2}=\bigcup_{(z_1,z_2)\in\mathcal{K}_{1,0}}\left(\{(z_1,z_2)\}\times\mathcal{K}'_{z_1,z_2}\right) \tag{3.69}$$

and take again

$$\mathcal{G}'=\bigcup_{(\bar x_1,\bar x_2)\in\mathcal{G}'_{1,0}}\left(\{(\bar x_1,\bar x_2)\}\times\mathcal{G}'_{\bar x_1,\bar x_2}\right).$$

¿From the preceding, since $\ell_1\ell_2\le\min\{m_1m_2,N^{1/2}\}$ in (3.63) and (3.66), the factor in the moment bound (3.59) becomes now

$$Cq\psi(M_1M_2,\,\delta_0,\,K_0)\cdot\psi\bigg(\min\{m_1m_2,N^{1/2}\},\,\frac{\delta_1}{(\log\frac{L}{\delta_1})^2},\,\delta_1^{-3}(\log L)^2L\bigg).$$

Thus in (3.4), $N'=M_1M_2$, $N''=\min\{m_1m_2,N^{1/2}\}$ satisfy (3.6) $\big($and may be increased to satisfy also the lower bound in (3.5)$\big)$.

Also

$$\delta_0\cdot\frac{\delta_1}{(\log\frac{L}{\delta_1})^2}\overset{(3.51)}{>}c\frac{\delta}{(\log\frac{K}{\delta})^6}$$

which is condition (3.7).

Taking $K'=K_0, K''=\delta_1^{-3}(\log L)^2L$, (3.59) implies

$$K'\cdot K''<\delta^{-6}(\log K)^{17}K$$

and hence (3.8) holds.

¿From (3.65), (3.67), (3.69)

$$\begin{aligned}|\mathcal{G}'_{\bar x_1,\bar x_2}|&>\left\{1+\frac{\delta_1}{\delta_3(\log\frac{K}{\delta})^4}\frac{\bar{\bar m}_1\bar{\bar m}_2}{\ell_1\ell_2}\right\}\cdot\phi\bigg(\ell_1\ell_2,\,\delta_3,\,\delta_1^{-3}(\log L)^2L\bigg)\\&\overset{(3.62)}{>}\left\{1+\frac{\delta_1^7}{(\log\frac{K}{\delta})^8}\frac{m_1m_2}{\ell_1\ell_2}\right\}\cdot\phi\bigg(\ell_1\ell_2,\,\frac{\delta_1}{(\log\frac{K}{\delta})^2},\,\delta_1^{-3}(\log L)^2L\bigg).\end{aligned} \tag{3.70}$$

Define

$$N''=\min\left\{N^{1/2},\,1+\frac{\delta_1^7}{(\log\frac{K}{\delta})^8}m_1m_2\right\}. \tag{3.71}$$

Using property (3.1) of the function $\phi$, we verify that

$$(3.70) > \left(1 + \frac{\delta_1^7 m_1 m_2}{(\log \frac{K}{\delta})^8}\right) \frac{1}{N''} \cdot \phi\left(N'',\, \frac{\delta_1}{(\log \frac{K}{\delta})^2},\, \delta_1^{-3} (\log L)^2 L\right).$$

Hence, again by (3.1)

$$\begin{aligned} |\mathcal{G}'| &> \left(1 + \frac{\delta_1^7 m_1 m_2}{(\log \frac{K}{\delta})^8}\right) \frac{1}{N''} \cdot \phi(M_1 M_2,\, \delta_0,\, K_0) \cdot \phi\left(N'',\, \frac{\delta_1}{(\log \frac{K}{\delta})^2},\, \delta_1^{-3} (\log L)^2 L\right) \\ &> \phi(N',\, \delta_0,\, K_0) \cdot \phi\left(N'',\, \frac{\delta_1}{(\log \frac{K}{\delta})^2},\, \delta_1^{-3} (\log L)^2 L\right) \end{aligned} \tag{3.72}$$

denoting

$$N' = \left(1 + \frac{\delta_1^7 m_1 m_2}{(\log \frac{K}{\delta})^8}\right) \frac{M_1 M_2}{N''}. \tag{3.73}$$

Thus

$$N > N' N'' > \frac{\delta_1^7}{(\log \frac{K}{\delta})^8} \cdot \bar{\bar{N}}_1 \bar{\bar{N}}_2 > \delta^{12} \left(\log \frac{K}{\delta}\right)^{-25} N \Rightarrow (3.5)$$

and by (3.71)

$$N'' \leq N^{1/2}, N' < M_1 M_2 + \frac{\bar{\bar{N}}_1 \bar{\bar{N}}_2}{N^{1/2}} \overset{(3.60)}{<} C \delta^{-15} K^7 N^{1/2} \Rightarrow (3.6).$$

This proves Lemma 3.2.

## Section 4. Proof of Proposition 3, Part II

Recalling (2.13) and (2.14), we start from the pair of admissible functions

$$\phi(N, \delta, K) = \left(\frac{\delta}{K}\right)^C N \tag{4.1}$$

$$\psi(N, \delta, k) = \min(q^{(\frac{K}{\delta})^C},\, N^{1/2}) \tag{4.2}$$

($C =$ some constant). The $N^{\frac{1}{2}}$-bound in (4.2) is obtained from the obvious estimate

$$\left\| \sum_\alpha F_\alpha \left(\prod_{p \in \mathcal{P}_0} p^{\alpha_p} \theta\right) \right\|_q \leq \sum_\alpha \|F_\alpha\|_q \leq N^{1/2} \left(\sum \|F_\alpha\|_q^2\right)^{1/2} \tag{4.3}$$

(since $\alpha$ ranges in a set of size at most $N$).

Starting from (4.1), (4.2), we produce here a new pair of admissible functions by applying Lemma 3.2. The next statement does not yet imply Proposition 2 but displays already a much better behavior of $\psi$ in $K$.

**Lemma 4.3.** *Take*

$$\tilde{\phi}(N,\delta,K) = \left(\frac{\delta}{K}\right)^{C\log\log\frac{K}{\delta}} \cdot N \tag{4.4}$$

*and*

$$\tilde{\psi}(N,\delta,K) = q^{(\log\frac{K}{\delta})^{C/\gamma}} \cdot N^\gamma \tag{4.5}$$

*with $C$ an appropriate constant and $0<\gamma<1$ arbitrary.*

*Then $\tilde{\phi}, \tilde{\psi}$ are admissible.*

**Proof.**

We will make an iterated application of Lemma 3.2.

Fix $N,\delta,K$ and choose an integer $t$ of the form $2^\ell$ (to be specified). Starting from $\phi_0=\phi, \psi_0=\psi$, define recursively for $\ell'=0,1,\dots,\ell-1$

$$\phi_{\ell'+1}(N,\delta,K) = \min \phi_{\ell'}(N',\delta',K')\cdot\phi_{\ell'}(N'',\delta'',K'') \tag{4.6}$$

$$\psi_{\ell'+1}(N,\delta,K) = \max Cq\psi_{\ell'}(N',\delta',K')\cdot\psi_{\ell'}(N'',\delta'',K''). \tag{4.7}$$

where in (4.6), (4.7) the parameters $N',N'',\delta',\delta'',K',K''$ satisfy (3.4)-(3.7).

We evaluate $\tilde{\phi}=\phi_\ell, \tilde{\psi}=\psi_\ell$.

Iterating (4.6), we obtain clearly

$$\tilde{\phi}(N,\delta,K) = \prod_{\nu\in\{0,1\}^\ell} \phi(N_\nu,\delta_\nu,K_\nu) \tag{4.8}$$

where

$$(N_\nu)_{\nu\in\bigcup\limits_{\ell'\le\ell}\{0,1\}^{\ell'}}, \quad (\delta_\nu)_{\nu\in\bigcup\limits_{\ell'\le\ell}\{0,1\}^{\ell'}}, \quad (K_\nu)_{\nu\in\bigcup\limits_{\ell'\le\ell}\{0,1\}^{\ell'}}$$

satisfy by (3.4)-(3.7) the following constraints

$$N_\phi = N, \delta_\phi=\delta, K_\phi=K \tag{4.9}$$

$$N_\nu \ge N_{\nu,0}\cdot N_{\nu,1} \ge N_\nu\left(\frac{\delta_\nu}{\log K_\nu}\right)^{40} \tag{4.10}$$

$$N_{\nu,0}+N_{\nu,1} < \left(\frac{K_\nu}{\delta_\nu}\right)^{20} N_\nu^{1/2} \tag{4.11}$$

$$\delta_{\nu,0}\cdot\delta_{\nu,1} > \left(\log\frac{K_\nu}{\delta_\nu}\right)^{-6}\delta_\nu. \tag{4.12}$$

$$K_{\nu,0}\cdot K_{\nu,1} \le \delta_\nu^{-6}(\log K_\nu)^{20} K_\nu. \tag{4.13}$$

¿From (4.12), (4.13)

$$\log \frac{K_{\nu,0}}{\delta_{\nu,0}} + \log \frac{K_{\nu,1}}{\delta_{\nu,1}} < 8 \log \frac{K_\nu}{\delta_\nu}$$

and iteration implies

$$\max_{\nu \in \{0,1\}^{\ell'}} \log \frac{K_\nu}{\delta_\nu} \leq \sum_{\nu \in \{0,1\}^{\ell'}} \log \frac{K_\nu}{\delta_\nu} < 8^{\ell'} \log \frac{K}{\delta}. \tag{4.14}$$

Iteration of (4.12) gives

$$\begin{aligned}
\prod_{\nu \in \{0,1\}^{\ell'}} \delta_\nu &> \prod_{\nu \in \{0,1\}^{\ell'-1}} \left( \log \frac{K_\nu}{\delta_\nu} \right)^{-6} \prod_{\nu \in \{0,1\}^{\ell'-1}} \delta_\nu \\
&\overset{(4.14)}{>} 8^{-3\ell' 2^{\ell'}} \left( \log \frac{K}{\delta} \right)^{-3 \cdot 2^{\ell'}} \prod_{\nu \in \{0,1\}^{\ell'-1}} \delta_\nu \\
&> 8^{-3(\ell' 2^{\ell'} + (\ell'-1) 2^{\ell'-1} + \cdots)} \left( \log \frac{K}{\delta} \right)^{-3(2^{\ell'} + 2^{\ell'-1} + \cdots)} \delta \\
&> 8^{-6\ell' 2^{\ell'}} \left( \log \frac{K}{\delta} \right)^{-6 \cdot 2^{\ell'}} \delta.
\end{aligned} \tag{4.15}$$

Next. iterate (4.13). Thus

$$\begin{aligned}
\prod_{\nu \in \{0,1\}^{\ell'}} K_\nu &\leq \prod_{\nu \in \{0,1\}^{\ell'-1}} \delta_\nu^{-6} (\log K_\nu)^{20} \cdot \prod_{\nu \in \{0,1\}^{\ell'-1}} K_\nu \\
&\overset{(4.14),(4.15)}{<} \left( 8^{-3\ell' 2^{\ell'}} \left( \log \frac{K}{\delta} \right)^{-3 \cdot 2^{\ell'}} \delta \right)^{-6} \left( 8^{\ell'} \log \frac{K}{\delta} \right)^{10 \cdot 2^{\ell'}} \left( \prod_{\nu \in \{0,1\}^{\ell'-1}} K_\nu \right) \\
&< 8^{56\ell' 2^{\ell'}} \left( \log \frac{K}{\delta} \right)^{56 \cdot 2^{\ell'}} \delta^{-6\ell'} K.
\end{aligned} \tag{4.16}$$

$\blacksquare$

By (4.10)

$$\prod_{\nu\in\{0,1\}^{\ell'}} N_\nu > \prod_{\nu\in\{0,1\}^{\ell'-1}} \left(\frac{\delta_\nu}{\log K_\nu}\right)^{40} \cdot \prod_{\nu\in\{0,1\}^{\ell'-1}} N_\nu$$

$$> 8^{-140\ell' 2^{\ell'}} \delta^{40} \left(8^{\ell'} \log \frac{K}{\delta}\right)^{-140\cdot 2^{\ell'}} \prod_{\nu\in\{0,1\}^{\ell'-1}} N_\nu$$

$$> 8^{-280\ell' 2^{\ell'}} \left(\log \frac{K}{\delta}\right)^{-280\, 2^{\ell'}} \delta^{40\ell'} N. \tag{4.17}$$

Substitution of (4.1) in (4.8) gives by (4.15), (4.16), (4.17)

$$\tilde\phi(N,\delta,K) \geq \prod_{\nu\in\{0,1\}^{\ell}} \left(\frac{\delta_\nu}{K_\nu}\right)^{C} N_\nu$$

$$> e^{-C\ell 2^\ell} \left(\log \frac{K}{\delta}\right)^{-C2^\ell} \delta^{C\ell} K^{-C} N$$

$$> t^{-Ct} \left(\log \frac{K}{\delta}\right)^{-Ct} (\delta^{C\log t}) K^{-C} N. \tag{4.18}$$

Similarly, we will iterate (4.7) with (possibly different) parameters $(N_\nu), (\delta_\nu), (K_\nu)$ still satisfying (4.9)-(4.17).

By (4.2)

$$\tilde\psi(N,\delta,K) = (Cq)^t \prod_{\nu\in\{0,1\}^{\ell}} \min\left(q^{(\frac{K_\nu}{\delta_\nu})^C}, N_\nu^{1/2}\right). \tag{4.19}$$

¿From (4.12)(which implies that $\delta_{\nu,0},\ \delta_{\nu,1} > \left(\log \frac{K_\nu}{\delta_\nu}\right)^{-6} \delta_\nu$) and (4.14) that

$$\delta_\nu > 8^{-6\ell^2} \left(\log \frac{K}{\delta}\right)^{-6\ell} \delta \tag{4.20}$$

and from (4.13) (which implies that $K_{\nu,0}, K_{\nu,1} \leq \delta_\nu^{-6} (\log K_\nu)^{20} K_\nu$), (4.14), and (4.20) that

$$K_\nu < 8^{37\ell^3} \left(\log \frac{K}{\delta}\right)^{37\ell^2} \delta^{-6\ell} K. \tag{4.21}$$

Hence from (4.11), (4.20), (4.21)

$$N_{\nu,0} + N_{\nu,1} < 8^{800\ell^3} \left( \log \frac{K}{\delta} \right)^{800\ell^2} \delta^{-800\ell} K^{20} N_\nu^{1/2}$$

$$N_\nu < \left( 8^{800\ell^3} \big( \log \frac{K}{\delta} \big)^{800\ell^2} \delta^{-800\ell} K^{20} \right)^{(1+\frac{1}{2}+\frac{1}{4}+\cdot)} N^{\frac{1}{2^\ell}}$$

$$< 10^{10^3\ell^3} \left( \log \frac{K}{\delta} \right)^{10^4\ell^2} \delta^{-10^4\ell} K^{40} N^{\frac{1}{t}} \text{ for } \nu \in \{0,1\}^\ell. \tag{4.22}$$

To bound (4.19), let $A$ be a number to be specified and partition

$$\{0,1\}^\ell = I \cup J \text{ with } I = \left\{ \nu \in \{0,1\}^\ell \middle| \frac{K_\nu}{\delta_\nu} \le A \right\}.$$

It follows then from (4.19) and (4.22) that

$$(4.19) < q^{A^C t} \left[ C^{\ell^3} \left( \log \frac{K}{\delta} \right)^{10^4\ell^2} \delta^{-10^4\ell} K^{40} N^{\frac{1}{t}} \right]^{|J|}. \tag{4.23}$$

Now, we will estimate $|J|$.

¿From (4.15), (4.16)

$$A^{|J|} \le \prod_{\nu \in \{0,1\}^\ell} \frac{K_\nu}{\delta_\nu} < 8^{62\ell t} \cdot \left( \log \frac{K}{\delta} \right)^{62t} \delta^{-7\ell} K. \tag{4.24}$$

Take

$$2^\ell = t \sim \log \frac{K}{\delta} \tag{4.25}$$

and fixing $0 < \gamma < 1$, take

$$\log A \sim \gamma^{-1} \log t. \tag{4.26}$$

It follows then from (4.24) that

$$|J| < \frac{10^3 t \cdot \log t}{\log A} < \gamma t. \tag{4.27}$$

Hence clearly from (4.23) that

$$\tilde{\psi}(N,\delta,K) < q^{A^C t} e^{t^2 \log t} N^\gamma < q^{(\log \frac{K}{\delta})^{C/\gamma}} \cdot N^\gamma \tag{4.28}$$

which is (4.5).

Substitution of (4.25) in (4.18) gives (4.4).

This proves Lemma 4.3.

**Section 5. Proof of Proposition 3, Part III**

We will first use Lemma 3.2 and Lemma 4.3 to show

**Lemma 5.1.** *Assume again the moment $q$ fixed. Given $0 < \tau, \gamma < \frac{1}{2}$, for $i = 1, 2, 3$, there are positive constants $A_i = A_i(\tau, \gamma), B_i = B_i(\tau, \gamma)$ such that taking $N$ sufficiently large*

$$\begin{cases} \phi(N, \delta, K) = K^{-A_1}\, \delta^{A_2 \log\log N}\, e^{A_3 (\log\log N)^2}\ N^{1-\tau} & (5.2) \\ \psi(N, \delta, K) = K^{B_1}\, \delta^{-B_2 \log\log N}\ e^{-B_3 (\log\log N)^2} N^{\gamma} & (5.3) \end{cases}$$

*is a pair of admissible functions.*

**Proof.**

We will proceed in 2 steps.

First, some

**Notation.** We use '$\ell\ell$' to denote '$\log\log$'.

It follows from Lemma 4.3 (by taking $\frac{\gamma}{4}$ and assuming $\frac{K}{\delta} < N$) that

$$\begin{cases} \phi(N, \delta, K) = \left(\frac{\delta}{K}\right)^{C_0 \ell\ell N} N & (5.4) \\ \psi(N, \delta, K) = \min\left\{ \exp\left[\log q \cdot \left(\log \frac{K}{\delta}\right)^{\frac{4C_0}{\gamma}}\right] \cdot N^{\frac{\gamma}{4}},\ N^{1/2}\right\} & (5.5) \end{cases}$$

are admissible.

First, fix a large integer $\bar{N}$ (depending on $\tau, \gamma$) and define

$$A_3 \sim A_2 \sim A_1 = C_0\, \ell\ell \bar{N} \tag{5.6}$$

($A_2, A_3$ will be specified later).

Thus the expression in (5.2) is at most

$$\left(\frac{\delta}{K}\right)^{C_0 \ell\ell \bar{N}} e^{C(\ell\ell \bar{N})^3} N^{1-\tau} < \left(\frac{\delta}{K}\right)^{C_0 \ell\ell N} N = (5.4),$$

if $\bar{N}$ is large enough and $N \leq \bar{N}, \log N \sim \log \bar{N}$.

Taking $N \leq \bar{N}$, then $(5.5) < N^{\frac{\gamma}{2}} < (5.3)$, provided

$$\log q \cdot \left(\log \frac{K}{\delta}\right)^{\frac{4C_0}{\gamma}} < \frac{\gamma}{4} \log N \tag{5.7}$$

and

$$e^{B_3 (\ell\ell N)^2} < N^{\frac{\gamma}{2}}. \tag{5.8}$$

If (5.7) does not hold, then for some $c > 0$,

$$\frac{K}{\delta} > e^{(\log N)^{c\gamma}}. \tag{5.9}$$

Thus if we take

$$B_3 \sim B_2 \sim B_1 = (\log \bar{N})^{1-c\gamma} \tag{5.10}$$

(also $B_2, B_3$ to be specified later). (5.9) implies that (5.3) $> N$ if $N \leq \bar{N}$ and (5.8) holds. From (5.10) and this choice of $B_3$, clearly

(5.8) holds for $N \leq \bar{N}$, $\log N \sim \log \bar{N}$. Thus with preceding choice of $A_1, A_2, A_3, B_1, B_2, B_3$, (5.2), (5.3) is admissible in the range $N \leq \bar{N}$, $\log N \sim \log \bar{N}$.

Next we use Lemma 3.2 to establish that (5.2), (5.3) are also admissible in the range $N > \bar{N}$. We will proceed by induction on the size of $N$. Obviously (5.3) $> N^{1/2}$ if $(\frac{K}{\delta})^{B_1} > N^{1/2}$. Hence we may assume

$$\frac{K}{\delta} < N^{10^{-6}}. \tag{5.11}$$

We want to reduce $N$ by Lemma 3.2. Thus we estimate

$$\phi(N', \delta', K') \cdot \phi(N'', \delta'', K'') \tag{5.12}$$

from below and

$$\psi(N', \delta', K') \cdot \psi(N'', \delta'', K'') \tag{5.13}$$

from above, where $N'N'', \delta', \delta'', K', K''$ satisfy (3.5)-(3.8). Hence for $N >> 0$,

$$N \geq N'N'' > N\left(\frac{\delta}{\log N}\right)^{40} \underset{(5.11)}{>} N^{\frac{99}{100}} \tag{5.14}$$

$$N' + N'' < \left(\frac{K}{\delta}\right)^{20} N^{1/2} \underset{(5.11)}{<} N^{\frac{11}{20}} \tag{5.15}$$

$$\delta'\delta'' > \frac{10^{36}\delta}{(\log N)^6} \tag{5.16}$$

$$K'K'' < \delta^{-6}(\log N)^{20} K. \tag{5.17}$$

Condition (5.15) reduces indeed $N$ to scale $N^{\frac{11}{20}}$ for which we assume (5.2), (5.3) admissible (notice that since $N > \bar{N}$, $\log N', \log N'' \gtrsim \log \bar{N}$). In what follows, the role of the additional technical factors in (5.2), (5.3) will become apparent.

Substituting (5.2) in (5.12), we get

$$(K'K'')^{-A_1}(\delta')^{A_2 \ell\ell N'}(\delta'')^{A_2\ell\ell N''}\ e^{A_3[\ell\ell N')^2+(\ell\ell N'')^2]}(N'N'')^{1-\tau}. \tag{5.18}$$

¿From (5.14), (5.15)

$$\begin{aligned} & N^{\frac{11}{25}} < N', N'' < N^{\frac{11}{20}} \\ \frac{99}{100}\ell\ell N < \ell\ell N - \log\frac{25}{11} & < \ell\ell N', \ell\ell N'' < \ell\ell N - \log\frac{20}{11}. \end{aligned} \tag{5.19}$$

¿From (5.14)-(5.17), (5.19)

$$\begin{aligned} (5.18) > \ & \delta^{6A_1}(\log N)^{-20A_1}K^{-A_1}\left[\frac{\delta}{(\log N)^6}\right]^{A_2(\ell\ell N - \log\frac{20}{11})} e^{\frac{19}{10}A_3(\ell\ell N)^2}N^{1-\tau}\left(\frac{\delta}{\log N}\right)^{40(1-\tau)} \\ > \ & K^{-A_1}\cdot\delta^{A_2\ell\ell N}\ e^{A_3(\ell\ell N)^2}N^{1-\tau}\cdot u\cdot v, \end{aligned}$$

where

$$u = (\log N)^{-20A_1-6A_2\ell\ell N-40}\ e^{\frac{9}{10}A_3(\ell\ell N)^2} \tag{5.20}$$

$$v = \delta^{6A_1-(\log\frac{20}{11})A_2+40}. \tag{5.21}$$

and each of the factors (5.20) and (5.21) will be at least 1 for suitable choices $A_1 < A_2 < A_3$ ($A_1 \sim A_2 \sim A_3$). Hence (5.18) still admits (5.2) as lower bound.

Similarly, substituting (5.3) in (5.13), we get

$$\begin{aligned} & (K'K'')^{B_1}(\delta')^{-B_2\ell\ell N'}(\delta'')^{-B_2\ell\ell N''}\ e^{-B_3[(\ell\ell N')^2+(\ell\ell N'')^2]}(N'N'')^{\gamma} \\ < \ & \delta^{-6B_1}(\log N)^{20B_1}K^{B_1}\left[\frac{(\log N)^6}{\delta}\right]^{B_2(\ell\ell N-\log\frac{20}{11})} e^{-\frac{19}{10}B_3(\ell\ell N)^2}\ N^{\gamma} \\ < \ & K^{B_1}\cdot\delta^{-B_2\ell\ell N}\ e^{-B_3(\ell\ell N)^2}N^{\gamma}\cdot u'\cdot v', \end{aligned}$$

where

$$u' = (\log N)^{20B_1+6B_2\ell\ell N}\cdot e^{-\frac{9}{10}B_3(\ell\ell N)^2} \tag{5.22}$$

$$v' = \delta^{-6B_1+B_2\log\frac{20}{11}}. \tag{5.23}$$

Again a choice $B_3 > B_2 > B_1$ ($B_3 \sim B_2 \sim B_1$) allows us to get (5.22) and (5.23) at most 1.

Thus (5.13) still satisfies (5.3).

This proves Lemma 5.1.

**Conclusion of the proof of Proposition 3**

Immediate from Lemma 5.1.

First, we may assume $K = K(\mathcal{G}) < N^{\frac{1}{\Lambda}}$ since (2.2) is otherwise obvious.

Apply Lemma 5.1 with $\tau, \gamma$ replaced by $\frac{\tau}{2}, \frac{\gamma}{2}$ and let

$$\Lambda = \frac{2A_1}{\tau} + A_2 + B_1 + \frac{2B_2}{\gamma}. \tag{5.24}$$

The choice of $\Lambda$ implies that $\Lambda > \frac{2A_1}{\tau}$, $\Lambda > B_1$, and $\frac{\Lambda}{2B_2}\gamma > 1$. In (5.2)

$$\phi(N, \delta, K) > \delta^{A_2 \ell\ell N} N^{1-\frac{\tau}{2}-\frac{A_1}{\Lambda}} > \delta^{\Lambda\ell\ell N} N^{1-\tau}. \tag{5.25}$$

In (5.3)

$$\psi(N, \delta, K) < K^{\Lambda} \cdot \delta^{-B_2\ell\ell N} N^{\frac{\gamma}{2}}. \tag{5.26}$$

If $\delta^{-B_2\ell\ell N} < N^{\frac{\gamma}{2}}$, then (5.26) $< K^\Lambda N^\gamma$. Otherwise in (2.1)

$$\delta^{\Lambda\ell\ell N} N^{1-\tau} < N^{-\frac{\Lambda}{2B_2}\gamma} N^{1-\tau} < 1,$$

hence the statement becomes trivial.

**Section 6. Remarks.**

(1) Going back to (1.12)-(1.15) and the proof of Lemma 5.1 and Proposition 3, an inspection of the argument shows that one may take $k(b) = C^{b^4}$ in the Theorem (for some constant $C$). We certainly did not try to proceed efficiently here.

(2) The proof of the Theorem shows in fact the following stronger statement:

For all $b \in \mathbb{Z}_+, \delta > 0$, there is $k = k(b, \delta) \in \mathbb{Z}_+$ such that whenever $A \subset \mathbb{Z}$, $|A| = N$ sufficiently large, then either $|kA| > N^b$ or $|A_1^{(k)}| > N^b$ for all $A_1 \subset A, |A_1| > N^\delta$.

(3) As in [Ch], our approach uses strongly prime factorization in $\mathbb{Z}$. Thus the argument at this point does not apply to subsets $A \subset \mathbb{R}$.

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